Complex Analysis: Final Exam

Aletta Jacobshal 01, Wednesday 31 January 2018, 18:30–21:30
Exam duration: 3 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of each answer sheet and on the envelope.
- Use the ruled paper for writing the answers and use the blank paper as scratch paper. After finishing put your answers in the envelope. **Do NOT seal the envelope!** You must return the scratch paper and the printed exam (separately from the envelope).
- Solutions should be complete and clearly present your reasoning. **When you use known results (lemmas, theorems, formulas, etc.) you must explicitly state and verify the corresponding conditions.**
- 10 points are "free". There are 6 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (15 points)

Evaluate

$$\mathrm{pv}\int_{-\infty}^{\infty} \frac{e^{-ix}}{x(x^2+1)}\,\mathrm{d}x$$

using the calculus of residues.

Solution

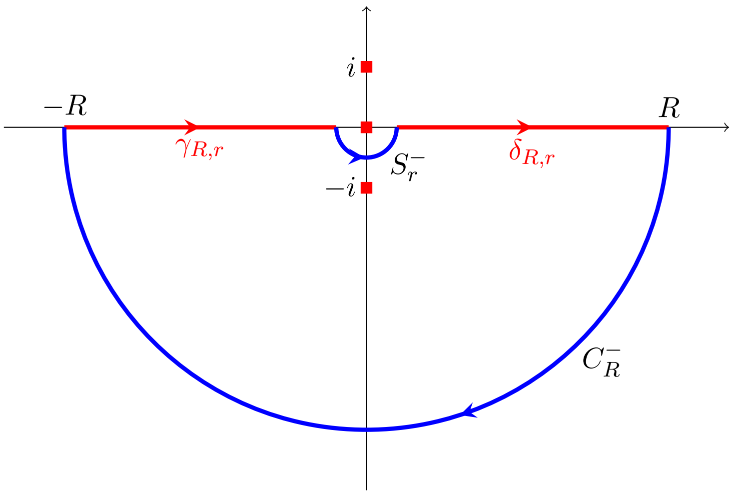
By definition,

$$\begin{aligned} I &= \mathrm{pv}\int_{-\infty}^{\infty} \frac{e^{-ix}}{x(x^2+1)}dx \\ &= \lim_{\substack{R\to\infty \\ r\to 0^+}} \left(\int_{-R}^{-r} \frac{e^{-ix}}{x(x^2+1)}dx + \int_{r}^{R} \frac{e^{-ix}}{x(x^2+1)}dx \right) \\ &= \lim_{\substack{R\to\infty \\ r\to 0^+}} I_{R,r}. \end{aligned}$$

To compute this integral we consider the closed contour

$$C_{R,r} = \gamma_{R,r} + S_r^- + \delta_{R,r} + C_R^-,$$

shown below.



We have

$$I_{R,r} = \int_{-R}^{-r} \frac{e^{-ix}}{x(x^2+1)}dx + \int_r^R \frac{e^{-ix}}{x(x^2+1)}dx$$
$$= \left(\int_{\gamma_{R,r}} + \int_{\delta_{R,r}}\right) f(z)dz,$$

where

$$f(z) = \frac{e^{-iz}}{z(z^2+1)}.$$

Therefore,

$$\int_{C_{R,r}} f(z)dz = I_{R,r} + \int_{S_r^-} f(z)dz + \int_{C_R^-} f(z)dz.$$

For $R > 1$ and $r < 1$ we have

$$\int_{C_{R,r}} f(z)dz = -2\pi i \operatorname{Res}(-i) = \pi i e^{-1},$$

where we used that

$$\operatorname{Res}(-i) = \lim_{z\to -i}(z+i)\frac{e^{-iz}}{z(z-i)(z+i)} = \lim_{z\to -i}\frac{e^{-iz}}{z(z-i)} = -\frac{1}{2}e^{-1},$$

and had to take a minus sign since $C_{R,r}$ is negatively oriented.

At the limit $r \to 0^+$ we have

$$\lim_{r\to 0^+}\int_{S_r^+} f(z)dz = \pi i \operatorname{Res}(0) = \pi i,$$

where we used that

$$\operatorname{Res}(0) = \lim_{z\to 0} z\frac{e^{-iz}}{z(z^2+1)} = \lim_{z\to 0}\frac{e^{-iz}}{z^2+1} = 1.$$

Moreover, since the degree of the numerator is 0 and the degree of the denominator is $3 \geq 0+1$, and since the exponential is of the form e^{-iz} and C_R^- lies in the lower half-plane we have from Jordan's lemma that

$$\lim_{R\to\infty} \int_{C_R^-} f(z)dz = 0.$$

Then taking the limits $R \to \infty$ and $r \to 0^+$ we get

$$\pi i e^{-1} = I + \pi i,$$

giving

$$I = (e^{-1} - 1)\pi i.$$

Question 2 (15 points)

Consider the polynomial $P(z) = z^4 + \varepsilon(z-1)$ where $\varepsilon > 0$. Show that if $\varepsilon < \dfrac{r^4}{1+r}$ then the polynomial P has four zeros inside the circle $|z| = r$.

Solution

The functions $f(z) = z^4$ and $P(z)$ are analytic on and inside the circle $|z| = r > 0$.

The number of zeros of z^4 inside this circle, counting multiplicity, is $N_0(f) = 4$.

Moreover, on the circle $|z| = r$ we have for $h(z) = \varepsilon(z+1)$ that

$$|h(z)| = |\varepsilon(z-1)| \leq \varepsilon(|z|+1) = \varepsilon(r+1) < r^4,$$

and

$$|f(z)| = |z^4| = r^4.$$

Therefore, on the circle $|z| = r$, we find

$$|h(z)| < r^4 = |f(z)|.$$

These facts mean that we can apply Rouché's theorem for $P = f + h$ to get

$$N_0(P) = N_0(f) = 4.$$

Question 3 (15 points)

Represent the function

$$f(z) = \frac{z-1}{z+1},$$

(a) (8 points) as a Taylor series around 0 and give its radius of convergence;

Solution

We have

$$\begin{aligned}\frac{z-1}{z+1} &= \frac{z-1}{1-(-z)} \\ &= (z-1)(1+(-z)+(-z)^2+(-z)^3+(-z)^4+\cdots) \\ &= (z-1)(1-z+z^2-z^3+z^4+\cdots) \\ &= (z-z^2+z^3-z^4+z^5+\cdots)-(1-z+z^2-z^3+z^4+\cdots) \\ &= -1+2z-2z^2+2z^3-2z^4+\cdots,\end{aligned}$$

where we used the geometric series which converges for $|z|<1$. The only singularity of $(z-1)/(z+1)$ is at $z=-1$ which is at a distance $|z|=1$ from 0. Therefore, the radius of convergence is 1.

(b) (7 points) as a Laurent series in the domain $|z|>1$.

Solution

Since $|z|>1$, that is $|1/z|<1$, we have

$$\begin{aligned}\frac{z-1}{z+1} &= \frac{1-\frac{1}{z}}{1+\frac{1}{z}} \\ &= \left(1-\frac{1}{z}\right)\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{z-1}{z+1} &= \left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right)-\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right) \\ &= 1-\frac{2}{z}+\frac{2}{z^2}-\frac{2}{z^3}+\frac{2}{z^4}-\frac{2}{z^5}+\cdots.\end{aligned}$$

Question 4 (15 points)

At which points is the function

$$f(z)=(1+i)x^2-(1-i)y^2,$$

differentiable and at which points is it analytic? Compute the derivative of $f(z)$ at the points where it exists.

Solution

We first bring the function into the standard form $u+iv$. We have

$$f(z)=(x^2-y^2)+i(x^2+y^2).$$

We check the partial derivatives, where we write $u(x,y)=x^2-y^2$ and $v(x,y)=x^2+y^2$. We have

$$\frac{\partial u}{\partial x}=2x, \qquad \frac{\partial u}{\partial y}=-2y,$$
$$\frac{\partial v}{\partial x}=2x, \qquad \frac{\partial v}{\partial y}=2y.$$

All partial derivatives exist and are continuous for all $x + iy \in \mathbb{C}$. Then the Cauchy-Riemann equations give $2x = 2y$ and $-2y = -2x$, that is, the function is differentiable at $z = x + iy$ only when $x = y$.

The function is nowhere analytic since the set where the function is differentiable contains no open sets.

The derivative at the points $z = x + iy$ with $x = y$ is given by

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + 2ix = 2(1+i)x.$$

Question 5 (15 points)

Consider the function

$$f(z) = \frac{e^z}{(z-1)^2}.$$

(a) (6 points) Determine the singularities of $f(z)$ and their type (removable, pole, essential; if pole, specify the order). *Hint: You can compute the Laurent series of the function but there are also other ways to determine the type of the singularity that do not require such computation.*

Solution

The function has a singularity at $z = 1$. To determine the type of the singularity we may work in one of the following ways.

Bounded limit. We notice that

$$\lim_{z\to 1}\left[(z-1)^2 \cdot \frac{e^z}{(z-1)^2}\right] = e,$$

implying (since the limit is bounded) that $z = 1$ is a pole of order 2.

Characterization of poles. We notice that

$$f(z) = \frac{g(z)}{(z-1)^2},$$

where $g(z)$ is an analytic function in a neighborhood of 1 (it is actually entire) and $g(1) = e \neq 0$. Therefore, $z = 1$ is a pole of order 2 for f.

Laurent series. The Taylor series for e^z at $z = 1$ is

$$\begin{aligned} e^z &= e \cdot e^{z-1} \\ &= e \cdot \left(1 + (z-1) + \frac{1}{2}(z-1)^2 + \cdots\right) \\ &= e + e(z-1) + \frac{e}{2}(z-1)^2 + \cdots . \end{aligned}$$

Therefore,

$$\frac{e^z}{(z-1)^2} = \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{e}{2} + \cdots ,$$

implying that $z = 1$ is a pole of order 2.

(b) (6 points) Show that $f(z)$ does not have an antiderivative in $\mathbb{C} \setminus \{1\}$. *Hint: Compute the integral of f along an appropriately chosen contour.*

Solution

If a function has an antiderivative in a domain D then its integral over any closed contour in D must be 0. Let C be a positively oriented circle centered at 1. We compute, using the generalized Cauchy formula, that

$$\int_C \frac{e^z}{(z-1)^2} dz = 2\pi i (e^z)'|_{z=1} = 2\pi i e \neq 0.$$

Therefore, the function does not have an antiderivative.

Alternatively, using the Laurent series from subquestion (a), we can compute the integral as

$$\int_C \frac{e^z}{(z-1)^2} dz = 2\pi i \operatorname{Res}(1) = 2\pi i e \neq 0.$$

(c) (3 points) Explain why $f(z)$ has an antiderivative in $\mathbb{C} \setminus L$ where $L = \{x \in \mathbb{R} : x \geq 1\}$.

Solution

The set $\mathbb{C} \setminus L$ is simply connected, the given function is analytic in this set, and these properties imply the existence of an anti-derivative since then all loop integrals of the function in the given set vanish.

Question 6 (15 points)

Consider the function

$$u(x, y) = e^x \cos y,$$

(a) (7 points) Prove that the function is harmonic in $\mathbb{R}^2$.

Solution

We compute

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y,$$

and

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y, \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y.$$

The second order partial derivatives are continuous on $\mathbb{R}^2$ and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0.$$

Therefore, the function is harmonic.

(b) (8 points) Find a harmonic conjugate of $u(x, y)$.

Solution

A harmonic conjugate of u satisfies the Cauchy-Riemann equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin y, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos y.$$

Integrating the first equation gives

$$v(x, y) = e^x \sin y + g(y).$$

Substituting into the second equation we find

$$e^x \cos y + g'(y) = e^x \cos y,$$

implying $g(y)$ is constant. Choosing $g(y) = 0$ we get the conjugate harmonic

$$v(x, y) = e^x \sin y.$$